Practical stability of positive fractional discrete-time linear systems

T. KACZOREK*
Faculty of Electrical Engineering, Białystok Technical University, 45D Wiejska St, 15-351 Białystok, Poland

Abstract. A new concept (notion) of the practical stability of positive fractional discrete-time linear systems is introduced. Necessary and sufficient conditions for the practical stability of the positive fractional systems are established. It is shown that the positive fractional systems are practically unstable if corresponding standard positive fractional systems are asymptotically unstable.

Key words: practical stability, fractional, positive, discrete-time, linear, system, stability test.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems in more complicated and less advanced. An overview of state of the art in positive systems theory is given in monographs [1, 2].

Mathematical fundamentals of fractional calculus are given in the monographs [3–6]. The fractional positive linear continuous-time and discrete-time systems have been addressed in [7–11]. The first monograph on analysis and synthesis of control systems with delays was the monograph published by Gorecki in 1989 [12]. Stability of positive 1D and 2D systems has addressed in [13–17] and the stability of positive fractional linear systems has been investigated in [18, 19]. The reachability and controllability to zero of positive fractional linear systems have been considered in [20–22]. The fractional order controllers have been developed in [23]. A generalization of the Kalman filter for fractional order systems has been proposed in [24]. Fractional polynomials and nD systems have been investigated in [25]. The notion of standard and positive 2D fractional linear systems has been introduced in [26, 27].

In this paper a new concept of the practical stability of positive fractional discrete-time linear systems will be introduced and necessary and sufficient conditions for the practical stability will be established.

The paper is organized as follows.

In Section 2 the basic definition and necessary and sufficient conditions for positivity and asymptotic stability of the linear discrete-time systems are introduced. In Section 3 the positive fractional linear discrete-time systems are introduced. The main results of the paper are given in Section 4, where

---

*e-mail: kaczorek@isep.pw.edu.pl

---
Theorem 2 [1, 13]. For the positive system (4) the following statements are equivalent:

1. The system is asymptotically stable.
2. Eigenvalues \( z_1, z_2, \ldots, z_n \) of the matrix \( A \) have moduli less than 1, i.e. \( |z_k| < 1 \) for \( k = 1, \ldots, n \).
3. \( \det[I_n - zA] \neq 0 \) for \( |z| \geq 1 \).
4. \( \rho(A) < 1 \) where \( \rho(A) \) is the spectral radius of the matrix \( A \) defined by \( \rho(A) = \max_{1 \leq k \leq n} |z_k| \).
5. All coefficients \( \hat{a}_i \), \( i = 0, 1, \ldots, n - 1 \) of the characteristic polynomial
   \[
   p_A(z) = \det[I_n z - A] = z^n + \hat{a}_{n-1}z^{n-1} + \cdots + \hat{a}_1 z + \hat{a}_0
   \]
of the matrix \( \hat{A} = A - I_n \) are positive.
6. All principal minors of the matrix
   \[
   \Delta = I_n - \hat{A} = \begin{bmatrix}
   \varpi_{11} & \varpi_{12} & \cdots & \varpi_{1n} \\
   \varpi_{21} & \varpi_{22} & \cdots & \varpi_{2n} \\
   \vdots & \vdots & \ddots & \vdots \\
   \varpi_{n1} & \varpi_{n2} & \cdots & \varpi_{nn}
   \end{bmatrix}
   \]  
   are positive, i.e.
   \[
   |\varpi_{11}| > 0, |\varpi_{12}| > 0, \ldots, \det \Delta > 0.
   \]
7. There exists a strictly positive vector \( \varpi > 0 \) such that
   \[
   [A - I_n]\varpi < 0.
   \]

Theorem 3 [2]. The positive system (4) is unstable if at least one diagonal entry of the matrix \( A \) is greater than 1.

3. Positive fractional systems

In this paper the following definition of the fractional discrete derivative
\[
\Delta^\alpha x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \quad 0 < \alpha < 1
\]
will be used, where \( \alpha \in R \) is the order of the fractional difference, and
\[
\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\
\frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \ldots
\end{cases}
\]
Consider the fractional discrete linear system, described by the state-space equations
\[
\Delta^\alpha x_{k+1} = A x_k + B u_k, \quad k \in Z_+, \quad y_k = C x_k + D u_k,
\]
where \( x_k \in \mathbb{R}^n, u_k \in \mathbb{R}^m, y_k \in \mathbb{R}^p \) are the state, input and output vectors and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m} \).

Using the definition (9) we may write the equations (11) in the form
\[
x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = A x_k + B u_k, \quad (12a)
\]
\[
y_k = C x_k + D u_k.
\]

Definition 2. The system (12) is called the (internally) positive fractional system if and only if \( x_k \in \mathbb{R}_+^n \) and \( y_k \in \mathbb{R}_+^p \), \( k \in Z_+ \) for any initial condition \( x_0 \in \mathbb{R}_+^n \) and all input sequences \( u_k \in \mathbb{R}_+^m, k \in Z_+ \).

Theorem 4. The solution of equation (12a) is given by
\[
x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} B u_i,
\]
where \( \Phi_k \) is determined by the equation
\[
\Phi_{k+1} = (A + I_n \alpha) \Phi_k + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1},
\]
with \( \Phi_0 = I_n \).
The proof is given in [8].

Lemma 1 [8]. If
\[
0 < \alpha \leq 1
\]
then
\[
(-1)^{i+1} \binom{\alpha}{i} > 0 \text{ for } i = 1, 2, \ldots
\]

Theorem 5 [8]. Let \( 0 < \alpha < 1 \). Then the fractional system (12) is positive if and only if
\[
A + I_n \alpha \in \mathbb{R}_+^n \times \mathbb{R}^{n \times m}, \quad B \in \mathbb{R}_+^m \times \mathbb{R}, \quad C \in \mathbb{R}_+^{p \times n}, \quad D \in \mathbb{R}_+^{p \times m}.
\]

4. Practical stability

From (10) and (16) it follows that the coefficients
\[
c_j = c_j(\alpha) = (-1)^j \binom{\alpha}{j}, \quad j = 1, 2, \ldots
\]
strongly decrease for increasing \( j \) and they are positive for \( 0 < \alpha < 1 \). In practical problems it is assumed that \( j \) is bounded by some natural number \( h \).

In this case the equation (12a) takes the form
\[
x_{k+h} = A_\alpha x_k + \sum_{j=1}^{h} c_j x_{k-j} + B u_k, \quad k \in Z_+,
\]
where
\[
A_\alpha = A + I_n \alpha.
\]

Note that the equations (19) and (12b) describe a linear discrete-time system with \( h \) delays in state.

Definition 3. The positive fractional system (12) is called practically stable if and only if the system (19), (12b) is asymptotically stable.
Defining the new state vector

\[
\tilde{x}_k = \begin{bmatrix}
x_k \\
x_{k-1} \\
\vdots \\
x_{k-h}
\end{bmatrix},
\]
we may write the equations (19) and (12b) in the form

\[
\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k, \quad k \in \mathbb{Z}_+,
\]

\[
y_k = \tilde{C}\tilde{x}_k + \tilde{D}u_k,
\]

where

\[
\tilde{A} = \begin{bmatrix}
A_0 & c_1I_n & c_2I_n & \cdots & c_{h-1}I_n & c_HI_n \\
I_n & 0 & 0 & \cdots & 0 & 0 \\
0 & I_n & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & I_n & 0
\end{bmatrix} \in \mathbb{R}^{\tilde{n} \times \tilde{n}},
\]

\[
\tilde{B} = \begin{bmatrix}
B \\
0 \\
\vdots \\
0
\end{bmatrix} \in \mathbb{R}^{\tilde{n} \times m}, \quad \tilde{C} = \begin{bmatrix} C & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{\tilde{n} \times \tilde{n}},
\]

\[
\tilde{D} = D \in \mathbb{R}^{\tilde{n} \times m}, \quad \tilde{n} = (1 + h)n.
\]

To test the practical stability of the positive fractional system (12) the conditions of Theorem 2 can be applied to the system (22).

**Theorem 6.** The positive fractional system (12) is practically stable if and only if one of the following condition is satisfied

1. Eigenvalues \(\tilde{z}_k, k = 1, \ldots, \tilde{n}\) of the matrix \(\tilde{A}\) have moduli less 1, i.e.

\[
|\tilde{z}_k| < 1 \quad \text{for} \quad k = 1, \ldots, \tilde{n}.
\]

2. det[\(z\tilde{n} - \tilde{A}\)] \(\neq 0\) for \(z\geq 1\).

3. \(\rho(\tilde{A}) < 1\) where \(\rho(\tilde{A})\) is the spectral radius of the matrix \(\tilde{A}\) defined by \(\rho(\tilde{A}) = \max_{1 \leq k \leq \tilde{n}} |\tilde{z}_k|\).

4. All coefficients \(\tilde{a}_i, i = 0, 1, \ldots, \tilde{n} - 1\) of the characteristic polynomial

\[
p_{\tilde{A}}(z) = \det[I_{\tilde{n}}(z + 1) - \tilde{A}] = z^{\tilde{n}} + \tilde{a}_{\tilde{n} - 1}z^{\tilde{n} - 1} + \cdots + \tilde{a}_1z + \tilde{a}_0
\]

of the matrix \([\tilde{A} - I_{\tilde{n}}]\) are positive.

5. All principal minors of the matrix

\[
[\tilde{A} - I_{\tilde{n}}] = \begin{bmatrix}
\tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1\tilde{n}} \\
\tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2\tilde{n}} \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{a}_{\tilde{n}1} & \tilde{a}_{\tilde{n}2} & \cdots & \tilde{a}_{\tilde{n}\tilde{n}}
\end{bmatrix},
\]

\[
[\tilde{a}_{11}] > 0, \quad [\tilde{a}_{11} \tilde{a}_{12} \cdots \tilde{a}_{1\tilde{n}}] > 0, \ldots, \det[I_{\tilde{n}} - \tilde{A}] > 0.
\]
In this case the characteristic polynomial (24) has the form
\[ p_\alpha(z) = \det[I_\alpha(z + 1) - \tilde{A}] = \]
\[
\begin{bmatrix}
  z + 0.4 & 1 & -1 \\
  -1 & z + 1 & 0 \\
  0 & -1 & z + 1
\end{bmatrix}
\]
\[ = z^3 + 2.4z^2 + 1.675z + 0.2125. \tag{32} \]

All coefficients of the polynomial (32) are positive and by Theorem 6 the system is practically stable.

Using (28) we obtain
\[
\text{Adj}[I_\alpha - \tilde{A}] = \text{Adj}
\begin{bmatrix}
  0.4 & -1 & 1 \\
  -1 & 1 & 0 \\
  0 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
  2.0625 \\
  0.6500 \\
  1.6125
\end{bmatrix}
\]

and the condition (28) is satisfied.

**Theorem 8.** The positive fractional system (12) is practically stable only if the positive system
\[ x_{k+1} = A_\alpha x_k, \quad k \in \mathbb{Z}_+ \tag{33} \]
is asymptotically stable.

**Proof.** From (26b) we have
\[ (A_\alpha - I_n)\bar{\pi}_0 + c_1\bar{\pi}_1 + ... + c_6\bar{\pi}_6 < 0. \tag{34} \]

Note that the inequality (34) may be satisfied only if there exists a strictly positive vector \( \bar{\pi}_0 \in \mathbb{R}^6_+ \) such that
\[ (A_\alpha - I_n)\bar{\pi}_0 < 0, \tag{35} \]
since \( c_1\bar{\pi}_1 + ... + c_6\bar{\pi}_6 > 0 \)

By Theorem 2 the condition (35) implies the asymptotic stability of the positive system (33).

From Theorem 8 we have the following important corollary.

**Corollary.** The positive fractional system (12) is practically unstable for any finite \( h \) if the positive system (33) is asymptotically unstable.

**Theorem 9.** The positive fractional system (12) is practically unstable if at least one diagonal entry of the matrix \( A_\alpha \) is greater than 1.

**Proof.** The proof follows immediately from Theorems 8 and 3.

**Example 2.** Consider the autonomous positive fractional system described by the equation
\[ \Delta^\alpha x_{k+1} = \begin{bmatrix}
  -0.5 & 1 \\
  2 & 0.5
\end{bmatrix} x_k, \quad k \in \mathbb{Z}_+ \tag{36} \]

for \( \alpha = 0.8 \) and any finite \( h \).
Practical stability of positive fractional discrete-time linear systems


